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Analytical solutions for unsteady free convection in porous media

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Abstract. Analytic solutions for two of the similarity cases identified by Johnson and Cheng (1978) for the unsteady free-convection boundary-layer flow over an impermeable vertical flat plate adjacent to a fluid saturated porous medium are given in the present paper. These are the solutions corresponding to an exponential ($e^{a^2 t}$) and a power-law (t^m) variation of the surface temperature, respectively. They represent exact solutions for doubly infinite plates and approximate solutions for semi-infinite plates. In the latter cases their validity is restricted to the so-called 'conduction regime' of the flow. It is shown that in the power law case, physical solutions only exist in the range $m > -1$ of the temperature exponent and they can be expressed in terms of Kummer's confluent hypergeometric functions. For $m \geq 0$ exponentially decaying unique solutions were found, while in the range $-1 < m < 0$ both exponentially and algebraically decaying multiple solutions occur. The origin of the multiple solutions as well as the feasibility conditions of all the above mentioned solutions is discussed in detail.

Key words: algebraic decay, multiple solutions, porous medium, unsteady free convection, vertical plate

1. Introduction

Convective flow in porous media occurs widely in natural phenomena and in industrial applications such as geothermal energy extraction, oil recovery, food processing, building insulations, heat-storage beds, dispersion of chemical contaminants in various processes in the chemical industry and in the environment, to name just a few. In many of these applications the porous medium is in contact with impermeable surfaces on which either natural or mixed convection flow can occur. Much of the work on this subject has recently been reviewed in [1–4].

A study of the literature shows that Johnson and Cheng [5] were the first who tried to determine the necessary and sufficient conditions under which similarity solutions exist for free-convection boundary layers adjacent to non-isothermal flat plates in porous media, including unsteady cases and those with ambient thermal stratification. Many attempts were made to find analytical and numerical solutions to the similarity cases identified by Johnson and Cheng since their paper [5] has been published. Much of the work on this topic has been reviewed by Pop *et al.* [6] and Bradean *et al.* [7].

The main issue of the present paper is to give exact analytic solutions for two of the similarity cases identified in [5] concerning the unsteady free-convection boundary-layer flow over an impermeable non-isothermal vertical flat plate embedded in a fluid-saturated porous medium. Our solutions are exact solutions for vertical doubly infinite plates (*i.e.*, plates without a leading edge) and approximate 'asymptotic solutions' for semi-infinite plates which, according to

Johnson and Cheng [5] are valid at 'sufficiently large distances' x from the leading edge of the plate, such that changes of the temperature and velocity fields due to variation of x can be neglected. The latter situation corresponds physically to the flow regime in which, compared to the heat diffusion, the convective heat transport becomes negligible and, as discussed by Vafai and Tien [8], the inertia effects on the fluid flow can also be neglected. It is worth noticing here that in the physically analogous case of the unsteady free convection of clear fluids, for doubly infinite plates several exact solutions were given by Schetz and Eichhorn already in the early sixties, [9].

As pointed out by Yang in the Discussion of paper [9], in the case of a semi-infinite plate the exact solutions of Schetz and Eichhorn become approximate solutions which are valid during a short time interval after the commencement of the free-convection flow as well as in the asymptotic range, *i.e.*, at large distances away from the leading edge (see also [10] and [11]). The common feature of these 'short time' (or 'switching on') and 'asymptotic' regimes is that, compared to the viscous and heat diffusion, the convection effects are negligible small. For this reason these heat and fluid-flow regimes are often referred to as the 'conduction regime'. In this respect the physical analogy between the free-convection flow in porous media and in clear fluids is quite a close one. However, as we are aware, in the case of unsteady free-convection flows over semi-infinite plates, no explicit and generally valid quantitative estimate is known (neither for clear fluids, nor for the unsteady flow in porous media), no matter how short the 'short time' is and how far away from the leading edge the 'asymptotic' range begins.

2. Basic equations

Consider a vertical impermeable flat plate adjacent to a fluid-saturated porous medium of constant ambient temperature, the plate being assumed to be non-isothermal. Following Johnson and Cheng [5] we write the basic continuity, Darcy and energy equations of the corresponding free-convection problem in the boundary-layer and Boussinesq approximation in the (dimensional) form

$$u_X + v_Y = 0, \quad u_Y = \frac{g\beta K}{\upsilon} T_y, \quad \sigma T_\tau + u T_X + v T_Y = \alpha T_{YY}. \tag{1}$$

Here X and Y are the Cartesian coordinates along and normal to the plate, respectively, τ is the time variable, u and v are the velocity components along the X- and Y-axes, T is the fluid temperature (considered to be in thermal equilibrium with the solid skeleton), K is the permeability of the porous medium, g is the acceleration due to gravity, $\alpha, \beta, \upsilon = \mu/\rho$ are the effective thermal diffusivity, thermal expansion coefficient, kinematic viscosity and $\sigma = (\rho C_p)_m/(\rho C_p)_f$ the capacity ratio of the porous medium (m) and the fluid (f). The subscripts X, Y, τ denote partial derivatives with respect to these variables.

The (dimensional) stream function Ψ, defined by $u = \Psi_Y$ and $v = -\Psi_X$ satisfies the continuity equation identically. Thus, by introducing the dimensionless coordinates x and y, time t, stream function ψ and temperature field Θ, according to the definitions of [5],

$$x = \frac{X}{L}, \;\; y = \frac{Y}{L}, \;\; t = \frac{\alpha}{\sigma L^2}\tau, \quad \psi = \frac{1}{\alpha}\Psi, \;\; \Theta = \frac{g\beta K L}{\upsilon\alpha}(T - T_\infty), \tag{2}$$

where L stands for a reference length, we may write the basic Equations (1) as

$$\psi_{yy} = \Theta_y, \qquad \Theta_t + \psi_y \Theta_x - \psi_x \Theta_y = \Theta_{yy} \tag{3}$$

A comprehensive discussion concerning the existence of similarity solutions of these equations along with the boundary conditions

$$\begin{aligned} &\psi_x = 0, \ \ \Theta = \Theta_w(x,t) \quad \text{on} \quad y = 0, \\ &\psi_y = 0, \ \ \Theta = \Theta_\infty(x,t) \quad \text{on} \quad y \to \infty \end{aligned} \tag{4}$$

was given by Johnson and Cheng [5].

In the present paper we are interested only in such solutions of the boundary-value problem (3), (4) which describe unsteady free-convection velocity- and temperature- boundary layers that do not depend on the wall coordinate $x \geq 0$. We also assume a stable environment, $\Theta_\infty(x,t) =$ const. and choose its temperature as the origin of the Θ-scale.

With these assumptions, the single non-vanishing component of the velocity field is given by

$$u = \psi_y = \Theta(y,t), \tag{5}$$

where $\Theta = \Theta(x,t)$ is the solution of the Fourier type equation

$$\Theta_t = \Theta_{yy} \tag{6}$$

along with the boundary conditions

$$\begin{aligned} &\Theta = \Theta_w(t) \quad \text{on} \quad y = 0, \\ &\Theta \to 0 \quad \text{as} \quad y \to \infty. \end{aligned} \tag{7}$$

The surface temperature $\Theta_w(t)$ is assumed to be positive for any $t \geq 0$ such that the solutions of the boundary-value problem (6), (7) will also be subjected to the additional physical requirement

$$\Theta(y,t) \geq 0 \quad \text{for any} \quad y, t \geq 0. \tag{8}$$

It is immediately seen that the energy equation (6), which is obtained by neglecting in the second Equation (3) all the streamwise derivatives, describes the ‘conduction regime’ of the flow in which, compared to heat convection, diffusion becomes the dominant heat-transport mechanism. As explained in the Introduction, in the case of doubly infinite plates this operation is exact, while for semi-infinite plates the problem (6), (7) yields approximate ‘short time’ and ‘asymptotic’ solutions, respectively.

3. Explicit similarity solutions

As shown in [5], the problem (3)–(4) admits asymptotic solutions with a wall temperature variation $\Theta_w(t)$ of exponential and of power-law type, respectively. In the present context these can easily be obtained from Equations (6) and (7) as follows.

The exponential solution can immediately be recovered as an elementary separable solution of the linear diffusion Equation (6). It is:

$$\Theta(y,t) = \Theta_0 \cdot \mathrm{e}^{-ay+a^2t}, \tag{9}$$

where Θ_0 and a are arbitrary positive constants (a^2 is the separation constant). This solution corresponds to a surface temperature variation of the form

$$\Theta_w(t) = \Theta_0 \cdot \mathrm{e}^{a^2 t}. \tag{10}$$

The power-law-type similarity solutions predicted by Johnson and Cheng [5] are in fact those which result from Equation (6) by the well-known Boltzmann transformation. They possess the functional structure

$$\Theta_w(t) = \Theta_w(t) \cdot \theta(\eta), \quad \eta = \frac{y}{2\sqrt{t + t_0}} \tag{11}$$

and correspond to a surface-temperature variation of the form

$$\Theta_w(t) = \Theta_0 \cdot (t + t_0)^m, \tag{12}$$

where Θ_0, t_0 and m are arbitrary real constants, $\Theta_0 > 0$, $t_0 \geq 0$. The similarity part $\theta = \theta(\eta)$ of the solution (11) satisfies the ordinary differential equation

$$\theta'' + 2\eta\theta' - 4m\theta = 0 \tag{13}$$

along with the boundary conditions

$$\theta(0) = 1, \quad \theta(\infty) = 0. \tag{14}$$

(The primes denote differentiation with respect to the Boltzmann similarity variable η).

The aim of the present paper is to complete the work reported in [5] with a detailed investigation of the boundary-value problem (13), (14). Its solutions will be given in an explicit analytical form and the origin of the multiple solutions (encountered for $m < 0$) will be discussed. In addition to the similar temperature profiles $\theta = \theta(\eta)$ the quantity of main physical interest is the (dimensionless) surface heat flux

$$q_w(t) = -\frac{\partial\Theta}{\partial y}(0, t) = -\frac{1}{2}\Theta_0(t + t_0)^{m-\frac{1}{2}}\theta'(0), \tag{15}$$

i.e., basically the surface temperature gradient $\theta'(0)$.

In order to be specific, we start the discussion with some special values of the parameter m.

3.1. Case $m = n/2, n = 0, 1, 2, \ldots$

The corresponding solution of this case can be given in terms of the repeated integrals of the error function (see [12, Chapter 7)]. It is

$$\theta(\eta) = 2^n\Gamma\left(\frac{n+2}{2}\right) \cdot \mathrm{i}^n\mathrm{erfc}\eta, \quad \theta'(0) = -2\Gamma\left(\frac{n+2}{2}\right) \Big/ \Gamma\left(\frac{n+1}{2}\right), \tag{16}$$

where Γ is the gamma function. This solution can also be expressed in terms of Kummer's confluent hypergeometric function U (see [12, Chapter 13]) as follows:

$$\theta(\eta) = \frac{\Gamma\left(\frac{n}{2} + 1\right)}{\sqrt{\pi}}\mathrm{e}^{-\eta^2}U\left(\frac{n+1}{2}, \frac{1}{2}, \eta^2\right). \tag{17}$$

We notice that for $m = n/2, n = 0, 1, 2, \ldots$ these solutions are the unique solutions of the problem. The corresponding surface heat flux (15) is always positive. Thus, in the case of a constant surface temperature ($m = n = 0$) we have

$$\theta(\eta) = \text{erfc}\eta, \quad \theta'(0) = -\frac{2}{\sqrt{\pi}}, q_w(x) = -\frac{\Theta_0}{\sqrt{\pi(t+t_0)}}. \tag{18}$$

As shown by Equation (15), the surface temperature variation with $m = +1/2$ (*i.e.*, $n = 1$) results in a constant surface heat flux. In this case we have:

$$\theta(\eta) = \mathrm{e}^{-\eta^2} - \sqrt{\pi} \cdot \eta \cdot \text{erfc}\eta, \quad \theta'(0) = -\sqrt{\pi}, \ q_w(t) = \frac{\sqrt{\pi}}{2}\Theta_0. \tag{19}$$

3.2. CASE $m = -1/2$

In this case a simple first integral exists, namely:

$$\theta' + 2\eta\theta = \text{const} \tag{20}$$

Thus, the general solution can easily be found by the method of variation of constants. It is

$$\theta(\eta) = C_1\mathrm{e}^{-\eta^2} + C_2F(\eta) \tag{21}$$

where C_1 and C_2 are constants of integration and F stands for Dawson's integral (see [12, Chapter 7]):

$$F(\eta) = \mathrm{e}^{-\eta^2}\int_0^\eta \mathrm{e}^{+z^2}\mathrm{d}z. \tag{22}$$

The first boundary condition (14) implies $C_1 = 1$, while the second condition (14) is satisfied automatically for any value of C_2 since:

$$\theta(\eta) \to C_1\mathrm{e}^{-\eta^2} + \frac{C_2}{\eta} \quad \text{as} \quad \eta \to \infty. \tag{23}$$

The additional condition (8) requires $C_2 \geq 0$, while the derivative of Equation (21) immediately shows that the constant C_2 is simply the slope of the solution curve $\theta(\eta)$ at the wall, $C_2 = \theta'(0)$. Thus, we have

$$\theta(\eta) = \mathrm{e}^{-\eta^2}\left(1 + \theta'(0) \cdot \int_0^\eta \mathrm{e}^{+z^2}\mathrm{d}z\right) \tag{24}$$

Therefore, our problem (13), (14) admits for $m = -1/2$ the one-parameter family of multiple solutions (24), the parameter being the (non-negative) slope of the solution curves at the wall, $\theta'(0) \geq 0$. The vanishing value $\theta'(0) = 0$ of this parameter corresponds to an adiabatic wall and to the exponentially decaying solution $\theta(\eta) = \exp(-\eta^2)$. We mention that this solution is also contained in Equation (16) for $n = -1$. The solutions corresponding to non-vanishing (positive) values of $\theta'(0)$ decay algebraically as $\theta'(0)/(2\eta)$ and are associated with a reversed wall heat flux (15).

As an illustration, in Figure 1 three members of this family of multiple solutions are shown. As will be seen below, the special cases discussed above already reveal all the main characteristics of the solutions of the boundary-value problem (13), (14). It should also be mentioned here that multiple (dual) solutions in steady mixed-convection boundary-layer flow over a vertical flat plate in a porous medium were first discovered by Merkin [13]. Ingham and Brown [14] have later shown that such solutions can also occur in the case of a free-convection boundary layer over a vertical flat plate whose surface temperature is a power-law function of the distance from the leading edge of the plate.

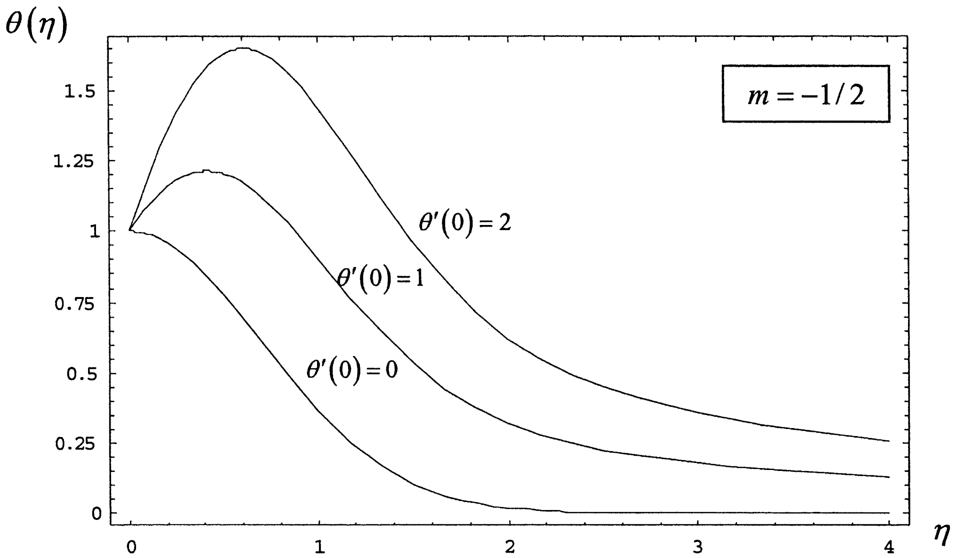

Figure 1. Three members of the family of multiple solutions (24) corresponding to $m = -1/2$ and $\theta'(0)$ 0, 1 and 2 respectively. The latter two decay algebraically as $\theta'(0)/\eta$ and the first one exponentially as $\exp(-\eta^2)$.

3.3. The general case

By the change of variables

$$\theta(\eta) = \mathrm{e}^{-z} \cdot w(z), \qquad z = \eta^2 \tag{25}$$

Equation (13) goes over in

$$z\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + \left(\frac{1}{2} - z\right)\frac{\mathrm{d}w}{\mathrm{d}z} - \left(m + \frac{1}{2}\right) w = 0, \tag{26}$$

which (in the notation of [12, Chapter 13]) coincides with Kummer's equation of the confluent hypergeometric functions $M(a, b, z)$ and $U(a, b, z)$ with $a = m + 1/2$ and $b = 1/2$. Thus, for an arbitrary m the general solution of Equation (13) can be expressed in terms of Kummer's M-function as follows ([12, Chapter 13]):

$$\theta(\eta) = C_1 M\left(-m, \frac{1}{2}, -\eta^2\right) + C_2 \eta M\left(\frac{1}{2} - m, \frac{3}{2}, -\eta^2\right). \tag{27}$$

The first boundary condition (14) implies again $C_1 = 1$, while the integration constant C_2 represents also in this case the slope $\theta'(0)$ of $\theta(\eta)$ at the wall, *i.e.*, $C_2 = \theta'(0)$. The asymptotic behaviour of (27) is given by:

$$\theta(\eta) \to \sqrt{\pi}\left[\frac{1}{\Gamma\left(m + \frac{1}{2}\right)} + \frac{\theta'(0)}{2\Gamma(m+1)}\right]\eta^{2m} \quad \text{as} \quad \eta \to \infty. \tag{28}$$

In this way, the following results emerge:

(i) For $m \geq 0$ the solution (27) of Equation (13) satisfies the second boundary condition (14) only if

$$\theta'(0) = -2\Gamma(m+1)\Big/\Gamma\left(m+\frac{1}{2}\right). \tag{29}$$

In this case $\theta'(0) < 0$ (direct wall heat flux), the solution (27) is unique and decays exponentially as $\eta \to \infty$. It also satisfies the additional condition (8) and can be expressed in terms of Kummer's U-function as follows:

$$\theta(\eta) = \frac{\Gamma(m+1)}{\sqrt{\pi}} e^{-\eta^2} \cdot U\left(m+\frac{1}{2}, \frac{1}{2}, \eta^2\right). \tag{30}$$

For $m = n/2$, $n = 0, 1, 2, \dots$ we recover from (29) and (30) the results (16) and (17).

(ii) For $m < 0$, the second boundary condition (14) is satisfied not only for $\theta'(0)$ given by Equation (29) but for any $\theta'(0)$. Hence, for $m < 0$ we obtain a one-parameter family of multiple solutions of the problem (13), (14), the parameter being the slope $C_2 = \theta'(0)$ of the solution curves at the wall. We underline again that the exponentially decaying solution (30) with $\theta'(0)$ given by Equation (29) also belongs to this family.

(iii) For $m \le -1$ no choice of the integration constant $\theta'(0)$ is possible, so that the additional condition (8) is satisfied for any $\eta \ge 0$. In Figure 2 such non-physical solutions of the boundary-value problem (13), (14) corresponding to $m = -1$ are shown.

(iv) In the range $-1 < m < 0$ the exponentially decaying solution (30) with $\theta'(0)$ given by Equation (29) satisfies the additional condition (8), while the algebraically decaying ones only do so when the wall slope $C_2 = \theta'(0)$ is chosen according to

$$\theta'(0) > -2\Gamma(m+1)\Big/\Gamma\left(m+\frac{1}{2}\right). \tag{31}$$

Thus, in the range $-1/2 < m < 0$ both solutions with $\theta'(0) < 0$ and $\theta'(0) > 0$ are possible, while in $-1 < m < 1/2$ all the multiple solutions correspond to $\theta'(0) > 0$, *i.e.*, to a reversed wall heat flux. For $m = -1/2$ and $C_2 = \theta'(0) \ge 0$ we recover in (27) the family of multiple solutions (24).

In Figure 3 the existence domain of the solutions of the problem (13), (14) is shown. The points of the thick 'border curve' specified by Equation (29) (which for large positive values of m approaches the asymptote $-2\sqrt{m}$) correspond to the exponentially decaying solutions (30). For $m \ge 0$ they represent the unique solution of the problem. For $-1 < m < 0$, however, the problem admits besides (30) also algebraically decaying multiple solutions (27) corresponding to values of $\theta'(0)$ lying above this border curve, *i.e.*, to values $\theta'(0)$ prescribed by inequality (31).

4. Feasibility

As we have seen in Section 2, the boundary-value problem (6), (7) admits for the surface temperature variation (11) unique similarity solutions for $m \ge 0$ and multiple ones for $-1 < m < 0$. The mathematical reason for the existence of multiple solutions is the simple fact that for $m < 0$ the second boundary condition (14) is automatically satisfied for any value of the integration constant $C_2 = \theta'(0)$ such that this condition becomes ineffective. Therefore, the basic question which arises is that of the feasibility (physically realistic) of all of these unique

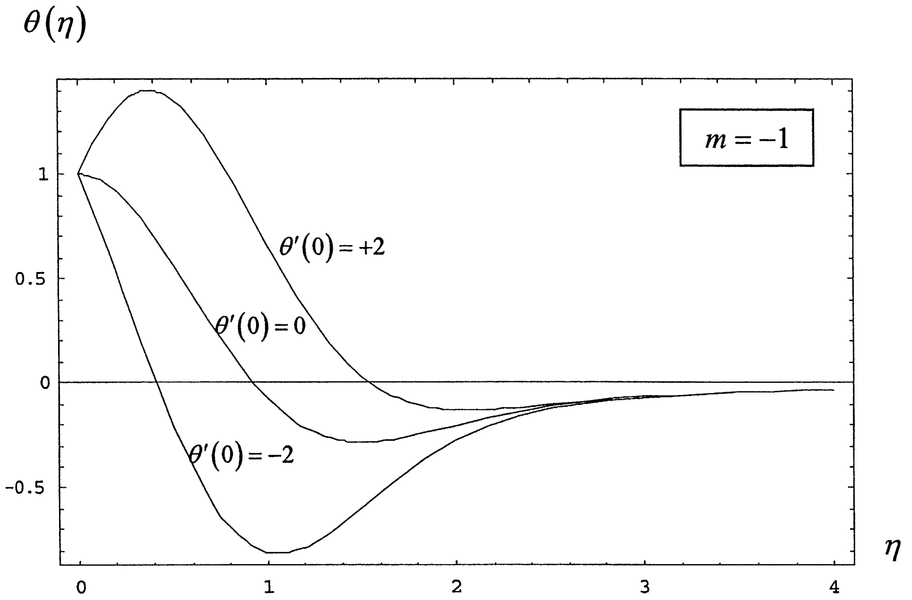

Figure 2. Non-physical solutions of the boundary value problem (13), (14) for $m = -1$ and $\theta'(0) = -2$, 0 and $+2$.

and multiple solutions of type (11) and (12). Obviously, the same question holds also for the exponential solution (9).

In order to elucidate this question we first notice that, strictly speaking, the boundary-value problem (6), (7) is ill-posed, since the parabolic character of Equation (6) requires to specify in addition to the boundary conditions also an initial condition at $t = 0$. Thus the well-posed counterpart of the problem (6), (7), involves instead of (7), the following initial and boundary conditions

$$\begin{aligned} t = 0: &\ \Theta = G(y) \quad \text{for} \quad y \geq 0, \\ t > 0: &\ \Theta = \Theta_w(t) \quad \text{on} \quad y = 0, \\ &\ \Theta \to 0 \quad \text{as} \quad y \to \infty, \end{aligned} \tag{32}$$

where the function $G = G(y)$ specifies the initial temperature distribution and at the same time, due to Equation (5), it also specifies the initial velocity distribution of the fluid.

Hence, our well-posed unsteady free-convection problem formally reduces to an unsteady heat-conduction problem in a semi-infinite solid extended to $y \geq 0$ and having the initial temperature distribution function $\Theta(y, 0) = G(y)$ for $y \geq 0$ and the prescribed surface-temperature variation $\Theta(0, t) = \Theta_w(t)$ for $t > 0$. The unique solution of this heat-conduction problem for arbitrary $G(y)$ and $\Theta_w(t)$ is well known and can be put in the form (see, *e.g.*, [15]):

$$\Theta(y, t) = \frac{e^{-\frac{y^2}{4t}}}{\sqrt{\pi t}} \int_0^\infty e^{-\frac{z^2}{4t}} \cdot \sinh\left(\frac{yz}{2t}\right) \cdot G(z) dz + \frac{2}{\sqrt{\pi}} \int_{\frac{y}{2\sqrt{t}}}^\infty e^{-z^2} \cdot \Theta_w\left(t - \frac{y^2}{4z^2}\right) dz \tag{33}$$

In case of the exponential solution (9), $\Theta_w(t)$ is given by Equation (10) and $G(y)$ is expressed as

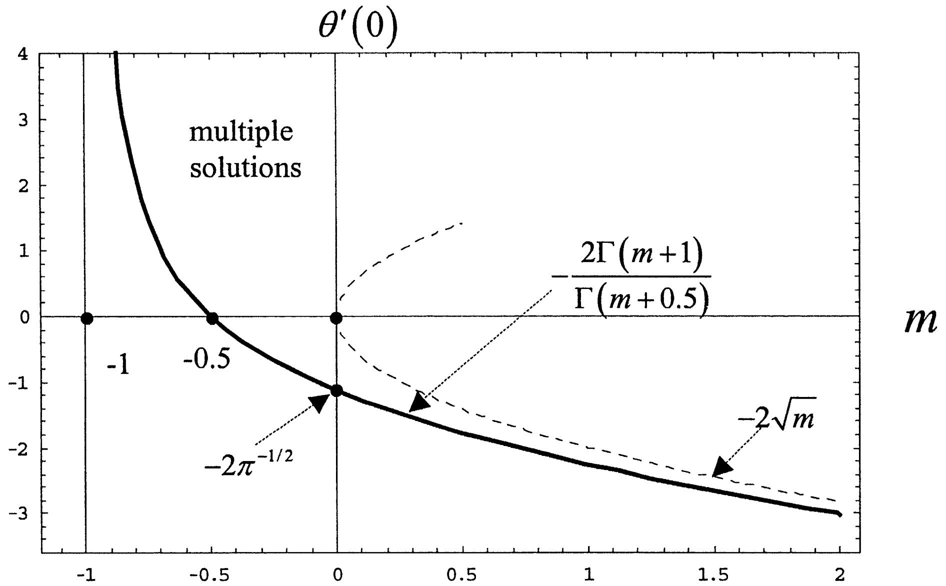

Figure 3. The existence domain of the solutions of problem (13), (14) consists of the thick 'border curve' for $m \geq 0$ and of this curve and all the points of the plane $(m, \theta'(0))$ above it for $-1 < m < 0$. For $m \leq -1$ no solutions satisfying the physical requirement (8) exist.

$$G(y) = \Theta_0 \cdot \mathrm{e}^{-ay}, \tag{34}$$

while in the power-law case (11), $\Theta_w(t)$ is given by Equation (12) and the initial temperature distribution is given according to Equations (12) and (27) by

$$G(y) = \Theta_0 \cdot t_0^m \left[M\left(-m, \frac{1}{2}, -\frac{y^2}{4t_0}\right) + \theta'(0)\frac{y}{2\sqrt{t_0}} M\left(\frac{1}{2} - m, \frac{3}{2}, -\frac{y^2}{4t_0}\right)\right]. \tag{35}$$

Therefore, the feasibility of the similarity solutions (9) and (27) is decided finally by the experimental realizability of the initial temperature (and, at the same, time velocity) configurations (34) and (35), respectively. Let us now discuss in this light the solutions (9) and (27) in detail.

4.1. Exponential solution (9)

First of all it can be shown that, as it should be, Equation (33) reproduces the solution (9) if $\Theta_w(t)$ and $G(y)$ are given by Equations (10) and (34), respectively. However, while it is easy to imagine that the exponential increase (10) of the temperature of the whole adjacent surface can actually be realized, it is hard to imagine that the distribution (34) of both the initial temperature and the initial velocity within the fluid is realizable experimentally. An easily realizable initial state is the static equilibrium state of an isothermal zero temperature quiescent fluid, *i.e.*, $\Theta(y, 0) = G(y) \equiv 0$ for any $y \geq 0$. The similarity solution (9) however, does not satisfy this initial condition. Consequently, it must be viewed as a solution which is non-realizable in practice.

On the other hand, the solution of Equation (6) corresponding to the exponential surface-temperature variation (10) and to the realistic initial condition $G(y) \equiv 0$ can not be obtained from the similarity transformation of Johnson and Cheng [5]. It results, however, from the general solution (33). It is clearly a non-similar solution and reads:

$$\Theta(y,t) = \frac{1}{2}\Theta_0 e^{at}\left[e^{-\sqrt{a}\cdot y}\cdot \mathrm{erfc}\left(\frac{y}{2\sqrt{t}} - \sqrt{at}\right) + e^{+\sqrt{a}\cdot y}\cdot \mathrm{erfc}\left(\frac{y}{2\sqrt{t}} + \sqrt{at}\right)\right]. \quad (36)$$

In contrast to the wall heat flux $q_w(t) = a\Theta_0 \exp(a^2 t)$ obtained from solution (9), the solution (36) yields the quite different result:

$$q_w(t) = \frac{\Theta_0}{\sqrt{\pi t}}\left(1 + \sqrt{\pi a t}\cdot e^{at}\cdot \mathrm{erfc}\sqrt{at}\right). \quad (37)$$

We notice that (36) and (37) are also valid for $a \le 0$, [16]. For $a = 0$, they correspond to Equation (18), respectively.

4.2. Power law solutions (27)

The situation is to some extent similar to that encountered in the exponential case (9). The initial temperature and velocity distribution $\Theta(y,0) = G(y)$ with $G(y)$ given by (35) can, in general, not be realized in practice. The static equilibrium state $G(y) \equiv 0$ represents again the realistic initial condition. For non-vanishing values of the constant t_0 the requirement $G(y) \equiv 0$ is certainly not satisfied. Let us examine therefore the limiting case $t_0 \to 0$. For $t_0 \to 0$, such that $y/\sqrt{t_0} \to \infty$ also for $y \to 0$, Equation (35) yields:

$$G(y) \to \sqrt{\pi}\Theta_0\cdot\left[\frac{1}{\Gamma\left(m+\frac{1}{2}\right)} + \frac{\theta'(0)}{2\Gamma(m+1)}\right]\left(\frac{y}{2}\right)^{2m}. \quad (38)$$

This means that, for the exponentially decaying solutions belonging to the border curve shown in Figure 3 for which Equation (29) holds and thus the square bracket in (38) vanishes identically, we have $G(y) \equiv 0$ for $t_0 = 0$. Therefore, the solutions

$$\Theta(y,t) = \Theta_0 \frac{\Gamma(m+1)}{\sqrt{\pi}} t^m \cdot \exp\left(-\frac{y^2}{4t}\right)\cdot U\left(m+\frac{1}{2}, \frac{1}{2}, \frac{y^2}{4t}\right) \quad (39)$$

are realizable for any $m > -1$. On the other hand, the algebraically decaying multiple solutions occurring for $-1 < m < 0$ and corresponding to values (31) of $\theta'(0)$, *i.e.*, the solutions for which the square bracket in (38) is not identically vanishing, are non-realistic. Again, it can be proven that solution (39) coincides with the general solution (33) for $G(y) \equiv 0$ and $\Theta_w(t) = \Theta_0 \cdot t^m$.

5. Conclusions

In the present paper two of the similarity cases identified by Johnson and Cheng [5] for the unsteady free-convection boundary-layer flow over an impermeable vertical flat plate adjacent to a fluid-saturated porous medium have been discussed. These are the solutions corresponding to an exponential (e^{at}) and a power-law (t^m) variation of the surface temperature, respectively. They represent exact solutions for doubly infinite plates and approximate ones for semi-infinite plates. In the latter cases their validity is restricted to the so-called ‘conduction regime’ of the flow.

The main issue in both of these cases was the uniqueness and the feasibility of the solutions obtained. We have shown that the non-uniqueness, *i.e.*, the occurrence of multiple solutions in the power-law case, is due to the fact that the boundary condition at infinity is satisfied for $m < 0$ automatically, such that one of the integration constants remains undetermined. This obviously leads to a one-parameter family of solutions of the boundary-value problem. It is worth mentioning here that this ineffectiveness of the boundary condition at infinity can frequently be encountered also in other segments of boundary-layer theory (see, *e.g.*, [17–19]).

On the other hand, the problem of feasibility is a more sensible one. It is decided in fact by the experimental realizability of the initial temperature and velocity distributions. Non-uniform initial temperature and velocity distributions are extremely difficult to be prepared, so that they must, in general, be considered as non-realizable in practice. Thus, the exponential case of the similarity solutions identified by Johnson and Cheng [5] must be considered as non-feasible. The same holds for the algebraically decaying solutions corresponding to the power-law variation of the surface temperature with the temperature exponent in the range $-1 < m < 0$. An easily realizable initial state is the isothermal state of a quiescent fluid. In this sense, the only feasible solutions described in this paper are the exponentially decaying ones given by (39) for $m > -1$.

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